TW

stichting
mathematisch
centrum

MC

TW

AFDELING TOEGEPASTE WISKUNDE TW 133/72 APRIL

T.H. KOORNWINDER

THE ADDITION FORMULA FOR JACOBI POLYNOMIALS
II THE LAPLACE TYPE INTEGRAL REPRESENTATION AND THE PRODUCT FORMULA

2e boerhaavestraat 49 amsterdam

Printed at the Mathematical Centre, 49, 2e Boerhaavestraat 49, Amsterdam.

The Mathematical Centre, founded the 11-th of February 1946, is a non-profit institution aiming at the promotion of pure mathematics and its applications. It is sponsored by the Netherlands Government through the Netherlands Organization for the Advancement of Pure Research (Z.W.O.), by the Municipality of Amsterdam, by the University of Amsterdam, by the Free University at Amsterdam, and by industries.

Abstract

This report contains the necessary preparations for a proof of the addition formula for Jacobi polynomials. Let Ω_{2q} be the unit sphere in a complex vector space C^q with hermitian inner product and with unitary group $U(q)$, then Ω_{2q} is a homogeneous space $U(q)/U(q-1)$. The class harm(m,n) of surface harmonics of type(m,n) on Ω_{2q}, introduced by Ikeda is invariant and irreducible under $U(q)$. Here, new proofs, based on orthogonality, are given for certain properties of these surface harmonics. Especially, it is proved that a zonal surface harmonic of type(m,n) equals

$$\text{const. } e^{i(m-n)\phi} (\cos\theta)^{|m-n|} P_{\min(m,n)}^{(q-2,|m-n|)} (\cos 2\theta)$$

for a suitable choice of coordinates in Ω_{2q}. The explicit expression of the kernel function for harm(m,n) gives the first stage of the addition theorem. Finally, a Laplace type integral representation and a product formula are derived for the zonal surface harmonics.

1. Introduction

This report is the second one in a series of papers dealing with the addition formula for Jacobi polynomials. In the first paper [11] the main results were announced. The present paper contains the necessary preparations for the proof of the addition formula. Furthermore, a Laplace type integral representation and a product formula are derived for the spherical functions on a certain homogeneous space. These results include the formulas (1) and (2) in [11] for Jacobi polynomials $P_n^{(\alpha,0)}(x)$. In a third paper [12] the proof of the addition formula will be completely settled.

In the classical theory of spherical harmonics functions are studied on the unit sphere in a real vector space, where the rotation group acts as a symmetry group. In a similar way, one can study functions on the unit sphere in a complex vector space with Hermitian inner product, where the group of unitary transformations acts as a symmetry group. This approach is due to Ikeda [8].

Let C^q be a complex vector space with unit sphere Ω_{2q} and with group of unitary transformations $U(q)$. Then, the subgroup of unitary transformations which leave one point of Ω_{2q} fixed is $U(q-1)$ and the sphere Ω_{2q} can be identified with the homogeneous space $U(q)/U(q-1)$. Functions on Ω_{2q} invariant under $U(q-1)$ are called zonal functions. It turns out that there exists an orthogonal decomposition for the function space $L^2(\Omega_{2q})$ into certain subspaces harm(m,n) (m,n integers ≥ 0), which are invariant and irreducible under $U(q)$. The functions in harm(m,n) are called surface harmonics of type(m,n). The zonal functions in harm(m,n) are:

$$(1.1) \quad \Phi(\theta,\phi) = \text{const. } e^{i(m-n)\phi} (\cos\theta)^{|m-n|} P_{\min(m,n)}^{(q-2,|m-n|)}(\cos 2\theta).$$

For this formula a suitable coordinate system of Ω_{2q} is chosen. In another terminology (cf. Helgason [7], p. 398) these functions are called the spherical functions on the homogeneous space $U(q)/U(q-1)$.

Functions on Ω_{2q} which are invariant under the unitary transformations $T = e^{i\phi} I$ can also be considered as functions defined on the complex projective space $SU(q)/U(q-1)$. Functions belonging to harm(n,n) satisfy this property and the only zonal surface harmonics of this type are

$$(1.2) \quad \Phi(\theta) = \text{const. } P_n^{(q-2,0)}(\cos 2\theta).$$

Surface harmonics on a complex projective space have been studied by Cartan ([2] and [3]). It is of interest to remark that the complex projective space $SU(q)/U(q-1)$ is a symmetric space (cf. Helgason [7]) in contrast with the sphere $U(q)/U(q-1)$. Anyhow, we need the larger space $U(q)/U(q-1)$ in order to obtain the full addition formula for Jacobi polynomials.

If in formula (1.1) $x = \cos\theta\cos\phi$ and $y = \cos\theta\sin\phi$ is taken then the functions Φ form an interesting class of orthogonal polynomials in the two variables x and y. In the case $q = 2$ these polynomials were noticed by Zernike and Brinkman [16], but the case of general q seems to be unnoticed until now. The product formula which is obtained for these polynomials in the present paper leads to a convolution structure generalizing the convolution structure for Jacobi series (cf. Gasper [6]). This convolution structure will be examined in a subsequent paper.

The methods and results in this and the following paper all have their analogues for classical spherical harmonics (cf. Erdélyi [5], Ch. 11, Müller [13] and Vilenkin [15], Ch. 9). The fundamental concepts in this paper are taken from Ikeda [8]. Some of the results in sections 2 and 3 were obtained earlier by Ikeda [8], Ikeda and Kayama [9] and Ikeda and Seto [10]. However, we preferred to make this paper rather self-contained for two reasons. Firstly, the publications [8] and [9] are rather unknown and difficult to obtain in this area of the world. Secondly, most of our proofs use orthogonality properties, while Ikeda obtains his results by solving certain differential equations. In our opinion, orthogonality methods are more elegant in the case of a compact space.

In section 2 of this report the fundamental concepts are presented and it is established that the spaces harm(m,n) are orthogonal to each other. In section 3 it is proved that the zonal functions in harm(m,n) are given by (1.1) and the addition formula is derived for an arbitrary orthonormal base of harm(m,n). The irreducibility and completeness of the spaces harm(m,n) are proved by using this addition formula. Section 4 deals with the Laplace type integral representation and product formula for the functions given in (1.1). The methods of proof are similar to those in Braaksma and Meulenbeld [1] and Dijksma and Koornwinder [4], respectively.

In the next report [12] of this series of papers a canonical orthonormal base for the space harm(m,n) will be constructed and, by using this base, the addition formula given in section 3 of this report will be specified. By a simple argument the addition formula for general $P_n^{(\alpha,\beta)}$ is then obtained and the product formula and Laplace type formula in the general case follow easily from this result (formulas (3), (2) and (1) in [11]).

Recently, R. Askey communicated to the author that a certain identity for Jacobi polynomials can be proved from (1) and (2) in [11]. By inverting Askey's argument the author was able to obtain the product formula (2) from the Laplace type formula (1). Earlier, Askey had already pointed out that (1) can be proved in an elementary way. Finally, the author derived the addition formula (3) from (2) and G. Gasper found formula (3) by another elementary approach. These elementary proofs of (1), (2) and (3), in which no group theoretical methods are used, will appear in several publications by Askey, Koornwinder and Gasper, respectively.

Acknowledgement

The author is due to Professor Richard Askey for suggesting him this problem. Most of the research presented here was done by the author during his stay at the Institute Mittag-Leffler in Djursholm, Sweden. The author is also due to Professor Lennart Carleson for his kind hospitality.

2. Preliminaries

In this section the definition is given of surface harmonics on the unit sphere in a complex vector space and a number of elementary properties are derived. Most of the ideas and results in this section are due to Ikeda [8] (part I).

Let C^q be a q-dimensional complex vector space with Hermitian inner product. If $z = (z_1, z_2, \ldots, z_q)$ and $w = (w_1, w_2, \ldots, w_q)$ are elements of C^q then the inner product is

$$(2.1) \quad (z,w) = z_1\bar{w}_1 + \ldots + z_q\bar{w}_q \ .$$

The group of linear transformations of C^q which leave this inner product invariant (the unitary transformations) is denoted by $U(q)$. $U(q)$ is a compact and connected Lie group.

Let Ω_{2q} be the unit sphere in C^q. The group $U(q)$ is a transitive transformation group of Ω_{2q}.

Definition. A function Φ on Ω_{2q} belongs to the class hom(m,n) if it is the restriction to Ω_{2q} of a polynomial

$$F(z,\bar{z}) \equiv F(z_1, z_2, \ldots, z_q, \bar{z}_1, \bar{z}_2, \ldots, \bar{z}_q)$$

which is homogeneous of degree m in its q complex variables z_k and homogeneous of degree n in its q complex variables $\bar{z}_k$.

In this definition, by the restriction of F to Ω_{2q} is meant that

$$(2.2) \quad \Phi(\xi) = F(\xi, \bar{\xi})$$

if $\xi \in \Omega_{2q}$ and $\bar{\xi}$ is the complex conjugate of ξ.

The variables z and $\bar{z}$ of the polynomial $F(z,\bar{z})$ may be formally considered as independent complex variables. But, since $F(z,\bar{z})$ is a polynomial, it is completely determined by its values in the case that z and $\bar{z}$ are complex conjugates. The mapping $F \to \Phi$ given by (2.2) is injective, for F can be expressed as

$$(2.3) \quad F(z,\bar{z}) = |z|^{m+n} \Phi\left(\frac{z}{|z|}\right)$$

if the vectors z and $\bar{z}$ are complex conjugates.

If $\Phi \in \mathrm{hom}(m,n)$ and if it is the restriction of the polynomial F then it is also the restriction of the polynomial

$$G(z,\bar{z}) \equiv (z_1\bar{z}_1+\ldots+z_q\bar{z}_q)\ F(z,\bar{z}).$$

Hence, there is the inclusion

$$(2.4) \quad \mathrm{hom}(m,n) \subset \mathrm{hom}(m+1,n+1).$$

A natural representation of the group U(q) in hom(m,n) is given by

$$(2.5) \quad (T\Phi)(\xi) \equiv \Phi(T^{-1}\xi) \qquad (\Phi\in\mathrm{hom}(m,n),\ T\in U(q),\ \xi\in\Omega_{2q}).$$

If Φ is the restriction of the polynomial F then $T\Phi$ is the restriction of the polynomial

$$(2.6) \quad TF(z,\bar{z}) \equiv F(T^{-1}z,\bar{T}^{-1}\bar{z}),$$

where $\bar{T}$ is the complex conjugate of T. Hence, the function $T\Phi$ also belongs to hom(m,n). The class hom(m,n) is invariant under unitary transformations.

For the particular unitary representation $T = e^{-i\phi}\ I$ we have

$$(2.7) \quad \Phi(e^{i\phi}\ \xi) = e^{i(m-n)\phi}\ \Phi(\xi) \qquad (\Phi\in\mathrm{hom}(m,n),\ \xi\in\Omega_{2q}).$$

This follows from (2.6) and the homogeneity of the polynomial F.

In order to obtain the dimension of hom(m,n) we note that a base for hom(m,n) is given by the restrictions to Ω_{2q} of the polynomials

$$z_1^{i_1} z_2^{i_2} \dots z_q^{i_q} \bar{z}_1^{j_1} \bar{z}_2^{j_2} \dots \bar{z}_q^{j_q}$$

where i_k and j_k are non-negative integers such that

$$i_1 + i_2 + \dots + i_q = m \quad \text{and} \quad j_1 + j_2 + \dots + j_q = n.$$

Hence, for the dimension $M(q;m,n)$ we have

$$M(q;m,n) = M(q;m)\, M(q;n)$$

where $M(q;l)$ is given by the partition formula

$$(1+x+x^2+\dots)^q = \sum_{l=0}^{\infty} M(q;l)\, x^l.$$

This can be rewritten as

$$\frac{1}{(1-x)^q} = \sum_{l=0}^{\infty} M(q;l)\, x^l \qquad (|x|<1).$$

It follows that

$$(2.8) \qquad M(q;m,n) \equiv \dim(\mathrm{hom}(m,n)) = \binom{m+q-1}{q-1}\binom{n+q-1}{q-1}$$

([8], p. 22) and

$$(2.9) \qquad \frac{1}{(1-x)^q(1-y)^q} = \sum_{m,n=0}^{\infty} M(q;m,n)\, x^m y^n \qquad (|x|<1,\ |y|<1).$$

Definition. A solid harmonic of type (m,n) is a polynomial

$$H(z,\bar{z}) \equiv H(z_1,z_2,\dots,z_q,\bar{z}_1,\bar{z}_2,\dots,\bar{z}_q)$$

which is homogeneous of degree m in its q complex variables $z_1,z_2,\dots,z_q$ and homogeneous of degree n in its q complex variables $\bar{z}_1,\bar{z}_2,\dots,\bar{z}_q$ and which satisfies

$$(2.10)\quad \left(\frac{\partial^2}{\partial z_1 \partial \bar{z}_1} + \dots + \frac{\partial^2}{\partial z_q \partial \bar{z}_q}\right) H(z,\bar{z}) = 0.$$

(The differentiation can be performed in a formal way.)

Definition. The class harm(m,n) consists of all functions on Ω_{2q} which are the restrictions to Ω_{2q} of solid harmonics of type (m,n). These functions are called surface harmonics of type (m,n).

It is evident from these definitions that

$$(2.11)\quad \text{harm}(m,n) \subset \text{hom}(m,n),$$

$$\text{harm}(m,0) = \text{hom}(m,0) \quad \text{and} \quad \text{harm}(0,n) = \text{hom}(0,n).$$

The differential operator in formula (2.10) is invariant under unitary transformations. Hence, a natural representation of U(q) in harm(m,n) is given by formula (2.5).

For a non-trivial example of a solid harmonic of type (m,n) consider the polynomial

$$(2.12)\quad H(z,\bar{z}) = (a_1 z_1 + \dots + a_q z_q)^m \, (b_1 \bar{z}_1 + \dots + b_q \bar{z}_q)^n.$$

If $\quad a_1 b_1 + \dots + a_q b_q = 0$

then H satisfies formula (2.10).

Let us write $z \in C^q$ as

$$z = x + iy = (x_1 + iy_1,\ x_2 + iy_2,\ \dots,\ x_q + iy_q).$$

Thus, C^q may be considered as a 2q-dimensional real vector space with coordinates $x_1, y_1, x_2, y_2, \dots, x_q, y_q$. If $z = x + iy$ and $z' = x' + iy'$ belong to C^q then the expression

$$\text{Re}[(z,z')] = \sum_{k=1}^{q} (x_k x'_k + y_k y'_k)$$

defines a real inner product on R^{2q}.
Unitary transformations of C^q leave this inner product invariant and act as orthogonal transformations of R^{2q}.

Let the function

$$H(z,\bar{z}) = H(x_1+iy_1, \ldots, x_q+iy_q, x_1-iy_1, \ldots, x_q-iy_q)$$

be a solid harmonic of type (m,n). It is clearly homogeneous of degree $m+n$ in its $2q$ variables $x_1,y_1,x_2,y_2,\ldots,x_q,y_q$ and, since

$$\sum_{k=1}^{q} \frac{\partial^2}{\partial z_k \partial \bar{z}_k} = \frac{1}{4} \sum_{k=1}^{q} \left(\frac{\partial^2}{\partial x_k^2} + \frac{\partial^2}{\partial y_k^2}\right) ,$$

it is also a harmonic function. Hence, the restriction of H to Ω_{2q} is a (complex-valued) spherical harmonic of degree $m+n$ in the classical sense.

The rotation-invariant surface element on the sphere Ω_{2q} will be denoted by $d\omega_{2q}$. For the total surface $\omega_{2q} = \int_{\Omega_{2q}} d\omega_{2q}$ it is well-known that

$$\omega_{2q} = \frac{2\pi^q}{(q-1)!} . \tag{2.13}$$

If Φ and Ψ are square-integrable functions on Ω_{2q} then their (hermitian) inner product is defined as

$$(\Phi,\Psi) = \int_{\Omega_{2q}} \Phi(\xi)\, \overline{\Psi(\xi)}\, d\omega_{2q}(\xi). \tag{2.14}$$

It follows that for $T \in U(q)$

$$(T\Phi,T\Psi) = (\Phi,\Psi). \tag{2.15}$$

Proposition 2.1. Let $S_1 \in \mathrm{harm}(m_1,n_1)$ and $S_2 \in \mathrm{harm}(m_2,n_2)$ and suppose that $(m_1,n_1) \neq (m_2,n_2)$. Then

$$\int_{\Omega_{2q}} S_1(\xi)\, \overline{S_2(\xi)}\, d\omega_{2q}(\xi) = 0.$$

Proof. If $m_1 + n_1 \neq m_2 + n_2$ then we use the orthogonality property for classical spherical harmonics of different degrees (cf. Müller [13], lemma 2).

If $m_1 + n_1 = m_2 + n_2$ then $m_1 - n_1 \neq m_2 - n_2$. For $T = e^{-i\phi} I$ we conclude from (2.15) and (2.7) that

$$(S_1,S_2) = (TS_1,TS_2) = e^{i((m_1-n_1)-(m_2-n_2)\phi} (S_1,S_2)$$

for all ϕ. Hence, $(S_1,S_2) = 0$.

This proposition was proved in [9], p. 99, by using an explicit base for harm(m,n).

Let e_k be the k-th unit vector in C^q. The subspace C^{q-k} is defined as the orthoplement of $e_1,e_2,\ldots,e_k$ and the subsphere Ω_{2q-2k} is defined as the intersection of Ω_{2q} and C^{q-k}. Let the subgroup $U(q-k)$ consist of all $T \in U(q)$ which leave the vectors $e_1,e_2,\ldots,e_k$ fixed. The group $U(q-k)$ is a transitive transformation group of Ω_{2q-2k} and leaves the rotation invariant measure $d\omega_{2q-2k}$ on Ω_{2q-2k} invariant.

Except for a set of lower dimension the elements $\xi \in \Omega_{2q}$ can be represented in a regular way as

$$(2.16) \quad \xi = \cos\theta\, e^{i\phi}\, e_1 + \sin\theta\, \xi'$$

with $\xi' \in \Omega_{2q-2}$, $0 < \theta < \pi/2$ and $\phi \in R$ mod.2π. In terms of these co-ordinates the line element on Ω_{2q} is

$$(2.17) \quad (d\xi)^2 = (d\theta)^2 + (\cos\theta)^2\,(d\phi)^2 + (\sin\theta)^2\,(d\xi')^2$$

and the surface element is

$$(2.18) \quad d\omega_{2q}(\xi) = \cos\theta\,(\sin\theta)^{2q-3}\,d\theta\,d\phi\,d\omega_{2q-2}(\xi').$$

Definition. For fixed $\alpha > -1$ and $\beta > -1$ Jacobi polynomials $P_n^{(\alpha,\beta)}$ are polynomials of degree n such that

(2.19) $$\int_{-1}^{+1} P_k^{(\alpha,\beta)}(x)\, P_l^{(\alpha,\beta)}(x)\, (1-x)^{\alpha}\, (1+x)^{\beta}\, dx = 0 \qquad \text{for } k \neq l$$

and

(2.20) $$P_n^{(\alpha,\beta)}(1) = \binom{\alpha+n}{n}.$$

We will often write

(2.21) $$R_n^{(\alpha,\beta)}(x) \equiv \frac{P_n^{(\alpha,\beta)}(x)}{P_n^{(\alpha,\beta)}(1)} .$$

Finally, the notation

$$m \wedge n \equiv \min(m,n) \quad \text{and} \quad m \vee n \equiv \max(m,n)$$

will be used.

3. Zonal surface harmonics and the addition formula

Firstly, zonal functions and their translates will be introduced.

Definition. Let Φ be a continuous function on Ω_{2q}. Φ is called a zonal function if

$$\Phi(T\xi) = \Phi(\xi) \qquad \text{for all } T \in U(q-1), \quad \xi \in \Omega_{2q}.$$

A zonal function Φ only depends on the inner product (ξ,e_1). For, using the coordinates defined by (2.16), we have

$$\Phi(\xi) = \Phi(\cos\theta\, e^{i\phi}\, e_1 + \sin\theta\, \xi') =$$

$$= \Phi(\cos\theta\, e^{i\phi}\, e_1 + \sin\theta\, e_2).$$

Here, a suitable transformation $T \in U(q-1)$ maps ξ' on e_2. It is clear that there is a function $f(z)$, continuous on the closed unit disk in the complex plane, such that

$$\Phi(\xi) = f(\cos\theta\, e^{i\phi}) = f((\xi,e_1)).$$

Definition. Let Φ be a zonal function and let

$$\Phi(\xi) = f((\xi,e_1)).$$

Then, for $\eta \in \Omega_{2q}$, the translate $\Phi(\xi,\eta)$ of $\Phi(\xi)$ is defined by

$$\Phi(\xi,\eta) \equiv f((\xi,\eta)).$$

It is obvious from this definition that

$$(3.1) \qquad \Phi(T\xi,T\eta) = \Phi(\xi,\eta) \qquad \text{for } T \in U(q)$$

and that

(3.2) $\Phi(\xi,e_1) = \Phi(\xi)$.

Conversely, if $\Phi(\xi,\eta)$ is a continuous function on $\Omega_{2q} \times \Omega_{2q}$ which satisfies (3.1) then the function $\Phi(\xi)$ defined by (3.2) is a zonal function which has $\Phi(\xi,\eta)$ as translate.

Lemma 3.1. (First stage of the addition theorem). Let V be a complex linear space of continuous functions on Ω_{2q} which is invariant under U(q) and has finite dimension $N > 0$. Then there is a unique non-zero zonal function Φ in V such that for every orthonormal base $\{S_1,S_2,\ldots,S_N\}$ of V

$$(3.3)\qquad \Phi(\xi,\eta) = \sum_{k=1}^{N} S_k(\xi)\,\overline{S_k(\eta)} \qquad (\xi,\eta\in\Omega_{2q}).$$

Furthermore, the identity

$$(3.4)\qquad \Phi(e_1)\,\omega_{2q} = N \quad \text{is valid.}$$

Proof. Let the function $\Phi(\xi,\eta)$ be defined by (3.3); it is independent of the choice of the orthonormal base. The unitary transformation T^{-1} maps the orthonormal base $\{S_1,\ldots,S_N\}$ onto the orthonormal base $\{T^{-1}S_1,\ldots,T^{-1}S_N\}$, hence identity (3.1) holds for $\Phi(\xi,\eta)$ and the function

$$\Phi(\xi) \equiv \Phi(\xi,e_1)$$

is zonal. It is apparent from (3.3) that this function $\Phi(\xi)$ is in V.

For the proof of (3.4) note that

$$\Phi(\xi,\xi) = \Phi(e_1,e_1) = \Phi(e_1) \quad \text{and}$$

$$\int_{\Omega_{2q}} \Phi(\xi,\xi)\, d\omega_{2q}(\xi) = \sum_{k=1}^{N} \int_{\Omega_{2q}} |S_k(\xi)|^2\, d\omega_{2q}(\xi) = N.$$

By (3.4) the function $\Phi(\xi)$ is a non-zero function.

Let V and Φ be as in lemma 3.1. From (3.3) we derive

$$\int_{\Omega_{2q}} |\Phi(\xi)|^2 \, d\omega_{2q} = \sum_{k=1}^{N} \sum_{l=1}^{N} \overline{S_k(e_1)}\, S_l(e_1) \int_{\Omega_{2q}} S_k(\xi)\, \overline{S_l(\xi)}\, d\omega_{2q} =$$

$$= \sum_{k=1}^{N} S_k(e_1)\, \overline{S_k(e_1)} = \Phi(e_1,e_1) = \Phi(e_1).$$

By combining this result with formula (3.4) it follows that

$$(3.5) \qquad \frac{1}{\omega_{2q}} \int_{\Omega_{2q}} \left|\frac{\Phi(\xi)}{\Phi(e_1)}\right|^2 d\omega_{2q}(\xi) = \frac{1}{N}\,.$$

Lemma 3.2. Let Φ be a zonal function belonging to hom(m,n). Let $\xi \in \Omega_{2q}$ be expressed as in formula (2.16). Then there exists a polynomial p of degree $\leq m \wedge n$ such that

$$(3.6) \qquad \Phi(\xi) = e^{i(m-n)\phi} (\cos\theta)^{|m-n|}\, p(\cos 2\theta).$$

Proof. Application of formula (2.7) gives

$$\Phi(\xi) = \Phi(\cos\theta\, e^{i\phi}\, e_1 + \sin\theta\, \xi') =$$

$$= e^{i(m-n)\phi}\, \Phi(\cos\theta\, e_1 + \sin\theta\, e^{-i\phi}\, \xi').$$

For every real ψ there exists a transformation $T \in U(q-1)$ which maps $e^{-i\phi}\xi'$ on $e^{i\psi}e_2$. It follows that

$$\Phi(\xi) = e^{i(m-n)\phi}\, \Phi(\cos\theta\, e_1 + \sin\theta\, e^{i\psi}\, e_2)$$

for all ψ, hence

$$\Phi(\xi) = e^{i(m-n)\phi}\, \frac{1}{2\pi} \int_0^{2\pi} \Phi(\cos\theta\, e_1 + \sin\theta\, e^{i\psi}\, e_2)\, d\psi.$$

Let Φ be the restriction of the polynomial $F(z,\bar{z})$ as given by formula (2.3), then

$$\Phi(\cos\theta\, e_1 + \sin\theta\, e^{i\psi}\, e_2) =$$

$$= F(\cos\theta,\ \sin\theta\, e^{i\psi},\ 0,\ \ldots,\ 0,\ \cos\theta,\ \sin\theta\, e^{-i\psi},\ 0,\ \ldots,\ 0) =$$

$$= \sum_{k=0}^{m} \sum_{l=0}^{n} a_{k,l} (\cos\theta)^{m-k} (\sin\theta\, e^{i\psi})^k (\cos\theta)^{n-l} (\sin\theta\, e^{-i\psi})^l =$$

$$= \sum_{k=0}^{m} \sum_{l=0}^{n} a_{k,l} (\cos\theta)^{m+n-k-l} (\sin\theta)^{k+l}\, e^{i(k-l)\psi}.$$

By using this expansion we obtain

$$\Phi(\xi) = e^{i(m-n)\phi} \frac{1}{2\pi} \int_0^{2\pi} \Phi(\cos\theta\, e_1 + \sin\theta\, e^{i\psi}\, e_2)\, d\psi =$$

$$= e^{i(m-n)\phi} \sum_{k=0}^{m\wedge n} a_{k,k} (\cos\theta)^{m+n-2k} (\sin\theta)^{2k} =$$

$$= e^{i(m-n)\phi} (\cos\theta)^{|m-n|} \sum_{k=0}^{m\wedge n} a_{k,k} (\cos\theta)^{2(m\wedge n-k)} (\sin\theta)^{2k} =$$

$$= e^{i(m-n)\phi} (\cos\theta)^{|m-n|} \sum_{k=0}^{m\wedge n} a_{k,k} \left(\frac{1+\cos 2\theta}{2}\right)^{m\wedge n-k} \left(\frac{1-\cos 2\theta}{2}\right)^{k}.$$

This proves the lemma.

Theorem 3.3. The function Φ on Ω_{2q} is a zonal function in harm(m,n) if and only if

$$(3.7)\qquad \Phi(\xi) = \text{const.}\ e^{i(m-n)\phi} (\cos\theta)^{|m-n|} P_{m\wedge n}^{(q-2,|m-n|)} (\cos 2\theta).$$

Proof. Let us write $k \equiv m-n$ and $l \equiv m\wedge n$. We suppose that k is a fixed integer and that l takes all values $0,1,2,\ldots$. Let Φ_l be a zonal function in harm(m,n). Then, by lemma 3.2, (3.8) $\quad \Phi_l(\xi) = e^{ik\phi}(\cos\theta)^{|k|}\; p_l(\cos 2\theta)$

where p_l is a polynomial of degree $\leq l$. For $l_1 \neq l_2$ the functions Φ_{l_1} and Φ_{l_2} are orthogonal (prop. 2.1). By substituting (3.7) and (2.18) we conclude from

$$\int_{\Omega_{2q}} \Phi_{l_1}(\xi)\, \overline{\Phi_{l_2}(\xi)}\, d\omega_{2q}(\xi) = 0 \qquad (l_1 \neq l_2)$$

that

$$\int_0^{\pi/2} p_{l_1}(\cos 2\theta)\, \overline{p_{l_2}(\cos 2\theta)}\, (\sin\theta)^{2q-3}\, (\cos\theta)^{2|k|+1}\, d\theta = 0 \qquad (l_1 \neq l_2).$$

By putting $x = \cos 2\theta$ it follows that

(3.9) $$\int_{-1}^{+1} p_{l_1}(x)\, \overline{p_{l_2}(x)}\, (1-x)^{q-2}\, (1+x)^{|k|}\, dx = 0 \qquad (l_1 \neq l_2).$$

Formula (2.12) gives an example of a non-zero function in harm(m,n). Hence, by lemma 3.1, there are non-zero zonal functions in harm(m,n) and, by formula (3.8) there exists a non-zero polynomial p_l of degree $\leq l$ for every integer $l \geq 0$. These polynomials p_l are determined by (3.9) up to a constant factor. Formula (2.9) shows that

$$p_l(x) = \text{const.}\; P_l^{(q-2,|k|)}(x).$$

This result, combined with (3.8), proves the theorem.

Theorem 3.3 is implicitly contained in reference [9]. The decomposition (3.10) below was first proved by Ikeda and Seto [10].

<u>Theorem 3.4</u>. There is an orthogonal decomposition of the space hom(m,n) given by

$$(3.10)\quad \hom(m,n) = \sum_{k=0}^{m\wedge n} {}^{\oplus}\mathrm{harm}(m-k,n-k).$$

The subspaces in this decomposition are invariant and irreducible under the group $U(q)$.

Proof. The formulas (2.11) and (2.4) show that harm(m-k,n-k) is contained in hom(m,n) and prop. 2.1 gives the orthogonality of the spaces harm(m-k,n-k). In order to prove the completeness of the decomposition we consider the orthoplement V of

$$\sum_{k=0}^{m\wedge n} {}^{\oplus}\mathrm{harm}(m-k,n-k) \qquad \text{in } \hom(m,n).$$

The space V is invariant under $U(q)$.
It will be shown that V has dimension zero by proving that every zonal function in V is zero (cf. lemma 3.1). Let

$$\Phi(\xi) = e^{i(m-n)\phi} (\cos\theta)^{|m-n|}\, p(\cos 2\theta)$$

a zonal function in V. The polynomial p has degree $\leq m\wedge n$. Φ is orthogonal to the zonal functions in harm(m-k,n-k). Hence, by theorem 3.3 and formula (2.18),

$$\int_{-1}^{+1} p(x)\, P_k^{(q-2,|m-n|)}(x)\, (1-x)^{q-2}\, (1+x)^{|m-n|}\, dx = 0$$

for $k = 0,1,\ldots,m\wedge n$. This proves that $p(x) \equiv 0$.

Finally, the irreducibility of the decomposition (3.10) will be proved. Suppose, on the contrary, that V is a non-trivial invariant subspace of harm(m,n). Then, the orthoplement W of V in harm(m,n) is also invariant under $U(q)$. By lemma 3.1, both disjoint subspaces V and W contain non-zero zonal functions Φ and Ψ, respectively. By theorem 3.3, these functions Φ and Ψ are equal up to a constant factor. This is a contradiction.

Corollary 3.5. Let $N(q;m,n)$ be the dimension of harm(m,n) and let $M(q;m,n)$ be the dimension of hom(m,n). Then

$$(3.11)\quad N(q;m,n) = M(q;m,n) - M(q;m-1,n-1) \qquad \text{for } m,n \neq 0,$$

$$N(q;m,0) = M(q;m,0) \quad \text{and}$$

$$N(q;0,n) = M(q;0,n).$$

Theorem 3.6. Let $N(q;m,n)$ be the dimension of harm(m,n). Then

$$(3.12)\quad N(q;m,n) = \frac{(m+n+q-1)\ (m+q-2)!\ (n+q-2)!}{m!\ n!\ (q-1)!\ (q-2)!}$$

and

$$(3.13)\quad \frac{1-xy}{(1-x)^q\ (1-y)^q} = \sum_{m,n=0}^{\infty} N(q;m,n)\ x^m\ y^n \qquad (|x|<1,\ |y|<1).$$

Proof. Formula (3.12) is obtained from (3.11) and (2.8). It follows from (3.11) that

$$\sum_{m,n=0}^{\infty} N(q;m,n)\ x^m\ y^n = (1-xy) \sum_{m,n=0}^{\infty} M(q;m,n)\ x^m\ y^n.$$

By combining this result with (2.9), formula (3.13) is proved.

Formula (3.12) can also be proved from formula (3.5) by using theorem 3.3. Ikeda [8] obtained formula (3.10) by algebraic considerations and then proved the irreducibility of harm(m,n) (our theorem 3.4) by applying Weyl's dimension formula for irreducible representations of SU(q).

We will need the identity

$$(3.14)\quad N(q;m,n) = \sum_{k=0}^{m} \sum_{l=0}^{n} N(q-1;k,l).$$

For the proof observe that

$$\sum_{m,n=0}^{\infty} N(q;m,n)\, x^m y^n = \frac{1-xy}{(1-x)^{q-1}\,(1-y)^{q-1}} \quad \frac{1}{(1-x)\,(1-y)} =$$

$$=\Big(\sum_{r,s=0}^{\infty} N(q-1;r,s)\, x^r y^s\Big)\Big(\sum_{i,j=0}^{\infty} x^i y^j\Big) =$$

$$= \sum_{m,n=0}^{\infty}\Big(\sum_{k=0}^{m}\sum_{l=0}^{n} N(q-1;k,l)\Big)\, x^m y^n.$$

The definition below is motivated by theorem 3.3.

Definition. For fixed $\alpha > -1$ functions $R_{m,n}^{(\alpha)}(1)$ on the unit disk D of the complex plane are defined by

$$(3.15)\quad R_{m,n}^{(\alpha)}(re^{i\phi}) \equiv R_{m\wedge n}^{(\alpha,|m-n|)}(2r^2-1)\; r^{|m-n|}\; e^{i(m-n)\phi}$$

(m and n integers ≥ 0).

For the argument of $R_{m,n}^{(\alpha)}$ we will write

$$z = x + iy = r\, e^{i\phi}.$$

It is easily seen that the functions $R_{m,n}^{(\alpha)}$ are polynomials of degree m+n in x and y, orthogonal on D with respect to the measure

$$(3.16)\quad (1-x^2-y^2)^{\alpha}\, dx\, dy = (1-r^2)^{\alpha}\, r\, dr\, d\phi.$$

There are N+1 different polynomials $R_{m,n}^{(\alpha)}$ of degree N. Hence, the polynomials $R_{m,n}^{(\alpha)}$ form a complete system of orthogonal polynomials in D with respect to the measure in (3.16). We collect the four properties which characterize the functions $R_{m,n}^{(\alpha)}$ in the following proposition.

Proposition 3.7. The functions $R_{m,n}^{(\alpha)}$ are characterized in a unique way by the following properties.

(i) $R_{m,n}^{(\alpha)}(x+iy)$ is a polynomial of degree m+n in x and y.

(ii) For every polynomial p(x,y) of degree < m+n

$$\iint_D R_{m,n}^{(\alpha)}(x+iy)\ \overline{p(x,y)}\ (1-x^2-y^2)^{\alpha}\ dx\ dy = 0.$$

(iii) $R_{m,n}^{(\alpha)}(e^{i\phi} z) = e^{i(m-n)\phi}\ R_{m,n}^{(\alpha)}(z)$.

(iv) $R_{m,n}^{(\alpha)}(1) = 1$.

For the case α = q-2 the functions $R_{m,n}^{(\alpha)}$ coincide with the zonal functions in harm(m,n) (cf. theorem 3.3). In Erdélyi ([5], §12.5, §12.6) a biorthogonal system of polynomials in two variables is given with respect to the measure (3.16). Zernike and Brinkman [16] have noticed the polynomials (3.15) in the case α = 0.

Finally, we will reformulate lemma 3.1 in the case that V = harm(m,n).

Thus, the second stage of the addition theorem is obtained. It is the analogue of theorem 4, p. 242 in Erdélyi [5].

Theorem 3.8. Let $S_1, S_2, \ldots, S_{N(q;m,n)}$ be an arbitrary orthonormal base of harm(m,n). Then

$$(3.17)\qquad \frac{N(q;m,n)}{\omega_{2q}}\ R_{m,n}^{(q-2)}((\xi,\eta)) = \sum_{k=0}^{N(q;m,n)} S_k(\xi)\ \overline{S_k(\eta)} \qquad (\xi,\eta\in\Omega_{2q}).$$

In our next report [12] theorem 3.8 will be specified by constructing a certain canonical orthonormal base for harm(m,n). This will be the third stage of the addition formula.

4. The Laplace type integral representation and the product formula for zonal surface harmonics.

The proofs of the two formulas mentioned in the title of this section have a technical detail in common which will be isolated in the following lemma.

Lemma 4.1. Let for every $\eta' \in \Omega_{2q-2}$ the function $\Phi(\theta,\phi,(\xi',\eta'))$ belong to harm(m,n) if considered as a function of $\xi = \cos\theta\, e^{i\phi}\, e_1 + \sin\theta\, \xi'$. Then

$$(4.1)\quad \frac{1}{\omega_{2q-2}}\int_{\Omega_{2q-2}} \Phi(\theta,\phi,(\xi',\eta'))\, d\omega_{2q-2}(\eta') = \text{const.}\; R_{m,n}^{(q-2)}(\cos\theta\, e^{i\phi}).$$

Furthermore,

$$(4.2)\quad \frac{1}{\omega_{2q-2}}\int_{\Omega_{2q-2}} \Phi(\theta,\phi,(\xi',\eta'))\, d\omega_{2q-2}(\eta') =$$

$$= \frac{q-2}{\pi}\int_0^1\int_0^{2\pi} \Phi(\theta,\phi,\, r\, e^{i\psi})\, r(1-r^2)^{q-3}\, d\psi\, dr \qquad \text{for } q = 3,4,5,\ldots$$

and

$$= \frac{1}{2\pi}\int_0^{2\pi} \Phi(\theta,\phi,e^{i\psi})\, d\psi \qquad \text{for } q = 2.$$

Proof. Let

$$S(\xi) \equiv \frac{1}{\omega_{2q-2}}\int_{\Omega_{2q-2}} \Phi(\theta,\phi,(\xi',\eta'))\, d\omega_{2q-2}(\eta').$$

Then $S \in$ harm(m,n). For $T \in U(q-1)$ we have

$$(T\xi',T\eta') = (\xi',\eta') \quad \text{and} \quad d\omega_{2q-2}(T\eta') = d\omega_{2q-2}(\eta')$$

hence $S(T\xi) = S(\xi)$ for $T \in U\,(q-1)$. Thus, formula (4.1) holds because S is a zonal function in harm(m,n) (cf. theorem 3.3 and formula (3.15)).

For the proof of (4.2) note that

$$S(\cos\theta\, e^{i\phi}\, e_1 + \sin\theta\, \xi') = S(\cos\theta\, e^{i\phi}\, e_1 + \sin\theta\, e_2) =$$

$$= \frac{1}{\omega_{2q-2}} \int_{\Omega_{2q-2}} \Phi(\theta,\xi,(e_2,\eta'))\, d\omega_{2q-2}(\eta').$$

By the substitutions

$$\eta' = r\, e^{-i\psi}\, e_2 + \sqrt{1-r^2}\, \eta'' \qquad (0<r<1,\ \psi\in R \bmod. 2\pi,\ \eta''\in\Omega_{2q-4})$$

and

$$d\omega_{2q-2}(\eta') = r(1-r^2)^{q-3}\, dr\, d\psi\, d\omega_{2q-4}(\eta'')$$

(cf. formulas (2.16) and (2.18)) formula (4.2) follows for $q > 2$. In the case $q = 2$ we use

$$\eta' = e^{-i\psi}\, e_2 \qquad \text{and} \qquad d\omega_2(\eta') = d\psi.$$

In the following only the formulas for $q > 2$ will be given. The easier analogues in the case $q = 2$ are left to the reader.

Theorem 4.2. For $q = 3,4,\ldots$ there is the integral representation

$$(4.3)\quad R^{(q-2)}_{m,n}(\cos\theta\, e^{i\phi}) = \frac{1}{\omega_{2q-2}} \int_{\Omega_{2q-2}} (\cos\theta\, e^{i\phi} + i\sin\theta\, (\xi',\eta'))^m \circ$$

$$\circ\ (\cos\theta\, e^{-i\phi} + i\sin\theta\, \overline{(\xi',\eta')})^n\, d\omega_{2q-2}(\eta') =$$

$$= \frac{q-2}{\pi} \int_0^1 \int_0^{2\pi} (\cos\theta\, e^{i\phi} + i\sin\theta\, r\, e^{i\psi})^m \circ$$

$$\circ\ (\cos\theta\, e^{-i\phi} + i\sin\theta\, re^{-i\psi})^n \circ r(1-r^2)^{q-3}\, d\psi\, dr.$$

Proof. Let in (2.12)

$$a = (a_1,\ldots,a_q) = e_1 + i\,\overline{\eta'} \quad \text{and}$$

$$b = (b_1,\ldots,b_q) = e_1 + i\,\eta'\ , \quad \text{where } \eta' \in \Omega_{2q-2}.$$

Then $a_1b_1 + \ldots + a_qb_q = 0$.
The restriction of the polynomial H in (2.12) to Ω_{2q} is

$$\Phi(\theta,\phi,(\xi',\eta')) \equiv (\cos\theta\, e^{i\phi} + i\sin\theta\,(\xi',\eta'))^m \circ$$

$$\circ\ (\cos\theta\, e^{-i\phi} + i\sin\theta\,\overline{(\xi',\eta')})^n.$$

For fixed η' this function belongs to harm(m,n). Application of lemma 4.1 gives formula 4.3, where the multiplicative constant is determined by by putting $\theta = 0$ and $\phi = 0$.

Theorem 4.2 suggests that a similar Laplace type formula holds in the case of non-integer q. In fact, it can be proved by application of the binomial formula that

$$(4.4)\qquad R_{m,n}^{(\alpha)}(\cos\theta\, e^{i\phi}) = \frac{\alpha}{\pi}\int_0^1\int_0^{2\pi} (\cos\theta\, e^{i\phi} + i\sin\theta\, r\, e^{i\psi})^m \circ$$

$$\circ\ (\cos\theta\, e^{-i\phi} + i\sin\theta\, r\, e^{-i\psi})^n\, r(1-r^2)^{\alpha-1}\, d\psi\, dr$$

for real $\alpha > 0$. In the special case that $m = n$ formula (4.4) can be written as

$$(4.5)\qquad R_n^{(\alpha,0)}(\cos 2\theta) =$$

$$= \frac{2\alpha}{\pi}\int_0^1\int_0^{\pi} ((\cos\theta)^2 - (\sin\theta)^2 r^2 + i\sin 2\theta\, r\cos\psi)^n \circ$$

$$\circ\ r(1-r^2)^{\alpha-1}\, d\psi\, dr, \qquad \alpha > 0.$$

For spherical functions Φ on a homogeneous space G/K there is the well-known product formula (see Helgason [7], p. 399)

$$(4.6) \qquad \Phi(x)\,\Phi(y) = \int_K \Phi(xky)\,dk,$$

where $x,y \in G$ and dk is the invariant measure on K. We will prove and reformulate this product formula for the functions $R_{m,n}^{(q-2)}$ in the case of the homogeneous space $U(q)/U(q-1)$.

Lemma 4.3. Let T_1 and $T_2 \in U(q)$ and put

$$(4.7) \qquad \begin{cases} T_1 e_1 = \cos\theta_1\, e^{i\phi_1} e_1 + \sin\theta_1\, \xi' & \text{and} \\[2ex] T_2^{-1} e_1 = \cos\theta_2\, e^{-i\phi_2} e_1 + \sin\theta_2\, \zeta'. \end{cases}$$

Let $d\mu$ the invariant measure on $U(q-1)$ with total measure 1. Then

$$(4.8) \qquad \int_{U(q-1)} R_{m,n}^{(\alpha)}((T_2TT_1e_1,e_1))\,d\mu(T) =$$

$$= \frac{1}{\omega_{2q-2}} \int_{\Omega_{2q-2}} R_{m,n}^{(\alpha)}(\cos\theta_1\cos\theta_2\, e^{i(\phi_1+\phi_2)} +$$

$$+ \sin\theta_1 \sin\theta_2(\xi',\eta'))\,d\omega_{2q-2}(\eta').$$

Proof. Substitution of (4.7) gives

$$(T_2TT_1e_1,e_1) = (T_1e_1,T^{-1}T_2^{-1}e_1) =$$

$$= (\cos\theta_1\, e^{i\phi_1} e_1 + \sin\theta_1\,\xi',\ \cos\theta_2\, e^{-i\phi_2} e_1 + \sin\theta_2\, T^{-1}\zeta')$$

$$= \cos\theta_1\cos\theta_2\, e^{i(\phi_1+\phi_2)} + \sin\theta_1\sin\theta_2(\xi',T^{-1}\zeta').$$

Hence, the left hand side of (4.8) is equal to

$$\int_{U(q-1)} R_{m,n}^{(\alpha)}(\cos\theta_1 \cos\theta_2 \, e^{i(\phi_1+\phi_2)} + \sin\theta_1 \sin\theta_2 \, (\xi', T^{-1}\zeta')\, d\mu(T).$$

For a function Φ on Ω_{2q-2} we have by the invariance of $d\mu$ that

$$\int_{U(q-1)} \Phi(T^{-1}\zeta')\, d\mu(T) = \int_{U(q-1)} \Phi(T^{-1}\eta')\, d\mu(T)$$

(ζ' and $\eta' \in \Omega_{2q-2}$). Therefore,

$$\int_{U(q-1)} \Phi(T^{-1}\zeta')\, d\mu(T) =$$

$$= \frac{1}{\omega_{2q-2}} \int_{T\in U(q-1)} \int_{\eta'\in\Omega_{2q-2}} \Phi(T^{-1}\eta')\, d\omega_{2q-2}(\eta')\, d\mu(T) =$$

$$= \frac{1}{\omega_{2q-2}} \int_{T\in U(q-1)} \int_{\eta'\in\Omega_{2q-2}} \Phi(\eta')\, d\omega_{2q-2}(\eta')\, d\mu(T) =$$

$$= \frac{1}{\omega_{2q-2}} \int_{\Omega_{2q-2}} \Phi(\eta')\, d\omega_{2q-2}(\eta').$$

Here, the second equality follows from the invariance of the measure $d\omega_{2q-2}$. By specifying the function Φ formula (4.8) is proved.

<u>Theorem 4.4</u>. For $q = 3,4,\ldots$ there are the product formulas

$$(4.9) \quad R_{m,n}^{(q-2)}((T_1e_1,e_1))\, R_{m,n}^{(q-2)}((T_2e_1,e_1)) =$$

$$= \int_{U(q-1)} R_{m,n}^{(q-2)}((T_2TT_1e_1,e_1))\, d\mu(T)$$

($T_1, T_2 \in U(q)$, $d\mu$ invariant measure on $U(q-1)$)

and

$$(4.10)\quad R^{(q-2)}_{m,n}(\cos\theta_1\, e^{i\phi_1})\, R^{(q-2)}_{m,n}(\cos\theta_2\, e^{i\phi_2}) =$$

$$= \frac{1}{\omega_{2q-2}} \int_{\Omega_{2q-2}} R^{(q-2)}_{m,n}(\cos\theta_1 \cos\theta_2\, e^{i(\phi_1+\phi_2)} + \sin\theta_1 \sin\theta_2\, (\xi',\eta'))$$

$$d\omega_{2q-2}(\eta') =$$

$$= \frac{q-2}{\pi} \int_0^1 \int_0^{2\pi} R^{(q-2)}_{m,n}(\cos\theta_1\, e^{i\phi_1} \cos\theta_2\, e^{i\phi_2} + \sin\theta_1 \sin\theta_2\, r\, e^{i\psi})$$

$$r(1-r^2)^{q-3}\, d\psi\, dr.$$

<u>Proof</u>. Let ξ and $\eta \in \Omega_{2q}$ and let us write

$$\xi = \cos\theta_1\, e^{i\phi_1}\, e_1 + \sin\theta_1\, \xi' \qquad (\xi' \in \Omega_{2q-2})$$

and

$$\eta = \cos\theta_2\, e^{-i\phi_2}\, e_1 + \sin\theta_2\, \eta' \qquad (\eta' \in \Omega_{2q-2}).$$

By theorem 3.3, by formula (3.15) and by the invariance of harm(m,n) the function $R^{(q-2)}_{m,n}((\xi,\eta))$ belongs to harm(m,n) as a function of ξ. Hence, for fixed θ_2 and ϕ_2, the function

$$\Phi(\theta_1,\phi_1,(\xi',\eta')) \equiv$$

$$\equiv R^{(q-2)}_{m,n}(\cos\theta_1 \cos\theta_2\, e^{i(\phi_1+\phi_2)} + \sin\theta_1 \sin\theta_2\, (\xi',\eta'))$$

belongs to harm(m,n) as a function of ξ.
Application of lemma 4.1 shows that there is a "constant" factor $c(\theta_2,\phi_2)$ such that the expression

$$c(\theta_2,\phi_2)\; R^{(q-2)}_{m,n}(\cos\theta_1\; e^{i\phi_1}) \qquad \text{is equal to}$$

the second and third expression in (4.10). By putting $\theta_1 = 0$ and $\phi_1 = 0$ we obtain that

$$c(\theta_2,\phi_2) = R^{(q-2)}_{m,n}(\cos\theta_2\; e^{i\phi_2}).$$

Finally, formula (4.9) follows from formula (4.10) and lemma 4.3.

The methods of proof in the theorems 4.2 and 4.4 should be compared with the methods used by Braaksma and Meulenbeld [1] and Dijksma and Koornwinder [4], respectively.

The analogues of (4.4) and (4.5) for the product formula are

$$(4.11)\quad R^{(\alpha)}_{m,n}(\cos\theta_1\; e^{i\phi_1})\; R^{(\alpha)}_{m,n}(\cos\theta_2\; e^{i\phi_2}) =$$

$$= \frac{\alpha}{\pi}\int_0^1\int_0^{2\pi} R^{(\alpha)}_{m,n}(\cos\theta_1\; e^{i\phi_1}\cos\theta_2\; e^{i\phi_2} + \sin\theta_1\sin\theta_2\; r\, e^{i\psi})\,\circ$$

$$\circ\; r\,(1-r^2)^{\alpha-1}\,d\psi\,dr$$

$$(\text{real } \alpha > 0)$$

and

$$(4.12)\quad R^{(\alpha,0)}_{n}(\cos 2\theta_1)\; R^{(\alpha,0)}_{n}(\cos 2\theta_2) =$$

$$= \frac{2\alpha}{\pi}\int_0^1\int_0^{\pi} R^{(\alpha,0)}_{n}(2(\cos\theta_1)^2\,(\cos\theta_2)^2 + 2(\sin\theta_1)^2\,(\sin\theta_2)^2\, r^2 +$$

$$+ \sin 2\theta_1\,\sin 2\theta_2\; r\cos\psi - 1)\; r\,(1-r^2)^{\alpha-1}\,d\psi\,dr$$

$$(\text{real } \alpha > 0).$$

Formula (4.12) is derived from (4.11) in the case m = n. Formula (4.11) will be proved in our next report [12] by integration of the addition formula for general α. Another method of proof proceeds by analytic continuation. Formula (4.11) holds for $\alpha = 1,2,\ldots$ (theorem 4.4) and therefore, by application of a theorem of Carlson (cf. Titchmarsh [14], p. 186), it holds for complex α with positive real part. See reference [4] for a similar application of Carlson's theorem. Finally, formula (4.11) can be obtained from (4.4) by using an identity for Jacobi polynomials due to Bateman. This elementary proof will be published in another place (cf. the remarks at the end of section 1).

References

[1] Braaksma, B.L.J. and B. Meulenbeld, Jacobi polynomials as spherical harmonics, Nederl. Akad. Wetensch. Proc. Ser. A71 = Indag. Math. 30 (1968), 384-389.

[2] Cartan, E., Sur la détermination d'un système orthogonal complet dans un espace de Riemann symétrique clos, Rend. Circ. Mat. Palermo 53 (1929), 217-252.

[3] Cartan, E., Leçons sur la géométrie projective complexe, Gauthier-Villars, Paris, 1931.

[4] Dijksma, A. and T.H. Koornwinder, Spherical harmonics and the product of two Jacobi polynomials, Nederl. Akad. Wetensch. Proc. Ser. A74 = Indag. Math. 33 (1971), 191-196.

[5] Erdélyi, A., et al., Higher Transcendental Functions, Vol. II, McGraw-Hill, New York, 1953.

[6] Gasper, G., Positivity and the convolution structure for Jacobi series, Ann. of Math. 93 (1971), 112-118.

[7] Helgason, S., Differential geometry and symmetric spaces, Academic Press, New York, 1962.

[8] Ikeda, M., On spherical functions for the unitary group (I,II,III), Mem. Fac. Engrg. Hiroshima Univ. 3 (1967), 17-75.

[9] Ikeda, M. and T. Kayama, On spherical functions for the unitary group (IV), Mem. Fac. Engrg. Hiroshima Univ. 3 (1967), 77-100.

[10] Ikeda, M. and N. Seto, On expansion theorems in terms of spherical functions for the unitary group (I), Math. Japon. 13 (1968), 149-157.

[11] Koornwinder, T.H., The addition formula for Jacobi polynomials, I Summary of results, Math. Centrum Amsterdam Afd. Toegepaste Wisk. Rep. TW 131 (1971), also to appear in Nederl. Akad. Wetensch. Proc. Ser. A.

[12] Koornwinder, T.H., The addition formula for Jacobi polynomials III, to appear, Math. Centrum Amsterdam, Afd. Toegepaste Wisk. Rep.

[13] Müller, C., Spherical harmonics, Lecture Notes in Math. 17, Springer-Verlag, Berlin, 1966.

[14] Titchmarsch, E.C., The theory of functions, Oxford University Press, second ed. 1939.

[15] Vilenkin, N.J., Special functions and the theory of group representations, Am. Math. Soc. Transl. of Math. Monographs, vol. 22, 1968.

[16] Zernike, F. und H.C. Brinkman, Hypersphärische Funktionen und die in sphärischen Bereichen orthogonalen Polynome, Nederl. Akad. Wetensch. Proc. 38 (1935), 161-173.